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Converses of Two Inequalities of Ky Fan, O. Taussky, and J. Todd

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In this paper we present simple new proofs of the inequalities

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos\frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2 \qquad (*)$$

which holds for all real numbers $a_0 = 0$, $a_1, ..., a_n$, $a_{n+1} = 0$, and

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos\frac{2\pi}{2n+1}\right) \sum_{k=1}^{n} a_k^2 \qquad (**)$$

which is valid for all real numbers $a_0 = 0$, $a_1, ..., a_n$. The constants $2(1 + \cos(\pi/(n+1)))$ and $2(1 + \cos(2\pi/(2n+1)))$ given in (*) and (**), respectively, are best possible.

1. INTRODUCTION

In 1955 Ky Fan, O. Taussky, and J. Todd [2] published a remarkable paper providing discrete analogues of several well-known integral inequalities. Among their results are the following two propositions.

THEOREM A. *If $a_1, ..., a_n$ are real numbers, where $a_0 = a_{n+1} = 0$, then*

$$2\left(1 - \cos\frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2 \leqslant \sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \qquad (1.1)$$

with equality holding if and only if $a_k = c \sin(k\pi/(n+1))$ $(k = 1, ..., n)$, where c is a real constant.

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And

THEOREM B. *If $a_1, \ldots, a_n$ are real numbers, where $a_0 = 0$, then*

$$2\left(1 - \cos \frac{\pi}{2n+1}\right) \sum_{k=1}^{n} a_k^2 \leqslant \sum_{k=1}^{n} (a_k - a_{k-1})^2 \tag{1.2}$$

with equality holding if and only if $a_k = c\ \sin(k\pi/(2n+1))$ $(k = 1, \ldots, n)$, where c is a real constant.

The two constants $2(1 - \cos(\pi/(n+1)))$ and $2(1 - \cos(\pi/(2n+1)))$ given in Theorem A and Theorem B, respectively, are best possible.

The "ingenious proof" [4, p. 126] presented by the authors for these results is based on an analysis of the characteristic values and vectors of Hermitean matrices. The main tool is an intriguing inequality of D. E. Rutherford who investigated the structure of Hermitean matrices "because of their great importance in a number of mathematical models of chemical and physical processes" [1, p. 183]. E. F. Beckenbach and R. Bellman mention that Theorem A and Theorem B as well as similar results are important for the numerical integration of differential equations, see [1, pp. 184–185].

It is natural to ask: Does there exist converse inequalities of (1.1) and (1.2)? This means: Is it possible to find values c_n and $\tilde{c}_n$ (which do not depend on the a_k's) such that

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \leqslant c_n \sum_{k=1}^{n} a_k^2$$

holds for real numbers $a_0 = 0, a_1, \ldots, a_n, a_{n+1} = 0$, and that

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \leqslant \tilde{c}_n \sum_{k=1}^{n} a_k^2$$

is valid for all real values $a_0 = 0, a_1, \ldots, a_n$?

An affirmative answer to this question was given by G. V. Milovanović and I. Ž. Milovanović [3] in 1982. Using techniques similar to those of Fan, Taussky, and Todd they proved (among other interesting propositions):

THEOREM 1. *For all real numbers $a_0 = 0, a_1, \ldots, a_n, a_{n+1} = 0$, we have*

$$\sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos \frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2. \tag{1.3}$$

Equality holds in (1.3) *if and only if*

$$a_k = c(-1)^{k-1} \sin \frac{k\pi}{n+1} \qquad (k = 1, ..., n), \tag{1.4}$$

where c is a real constant.

And

THEOREM 2. *For all real numbers* $a_0 = 0, a_1, ..., a_n$, *we have*

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos \frac{2\pi}{2n+1}\right) \sum_{k=1}^{n} a_k^2. \tag{1.5}$$

Equality holds in (1.5) *if and only if*

$$a_k = c(-1)^{k-1} \sin \frac{2k\pi}{2n+1} \qquad (k = 1, ..., n), \tag{1.6}$$

where c is a real constant.

Motivated to find "easy proofs of (the) hard inequalities" [4, p. 139] (1.1) and (1.2), R. M. Redheffer [4] presented in 1983 a very elegant elementary method to establish Theorem A and Theorem B. Using a modification of Redheffer's technique we will be able to give simple new proofs of (1.3) and (1.5).

2. New Proofs of Theorem 1 and Theorem 2

Before we establish the inequalities (1.3) and (1.5) we formulate and prove the following

LEMMA. *Let n be an integer with* $n > 1$ *and let t be a real number such that* $t \in (0, \pi/n)$. *Further let*

$$\mu = 2(1 + \cos t) \qquad \textit{and} \qquad \lambda_k = 1 + \frac{\sin(k+1)t}{\sin(kt)}, \qquad k = 1, ..., n.$$

Then we have for all real numbers $a_0 = 0, a_1, ..., a_n$,

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 + \lambda_n a_n^2 \leqslant \mu \sum_{k=1}^{n} a_k^2. \tag{2.1}$$

Equality holds in (2.1) *if and only if*

$$x_k a_k + y_{k-1} a_{k-1} = 0 \qquad \textit{for} \quad k = 2, ..., n,$$

where

$$x_k = (\mu - 1 - \lambda_k)^{1/2} \qquad \textit{and} \qquad y_{k-1} = (\lambda_{k-1} - 1)^{1/2}.$$

Proof. A simple calculation yields

$$x_k y_{k-1} = [(\mu - 1 - \lambda_k)(\lambda_{k-1} - 1)]^{1/2} = 1 \qquad \text{for} \quad k = 2, ..., n.$$

This leads to

$$\begin{aligned} 0 &\geqslant -[x_k a_k + y_{k-1} a_{k-1}]^2 \\ &= -x_k^2 a_k^2 - 2a_{k-1} a_k - y_{k-1}^2 a_{k-1}^2 \\ &= (a_k - a_{k-1})^2 + \lambda_k a_k^2 - \lambda_{k-1} a_{k-1}^2 - \mu a_k^2 \qquad \text{for} \quad k = 2, ..., n. \end{aligned}$$

If $k = 1$, then we have

$$0 = -[x_1 a_1 + y_0 a_0]^2 = (a_1 - a_0)^2 + \lambda_1 a_1^2 - \lambda_0 a_0^2 - \mu a_1^2.$$

Summing over $k = 1, ..., n$, we get

$$0 \geqslant \sum_{k=1}^{n} (a_k - a_{k-1})^2 + \lambda_n a_n^2 - \mu \sum_{k=1}^{n} a_k^2$$

with equality holding if and only if

$$x_k a_k + y_{k-1} a_{k-1} = 0 \qquad \text{for} \quad k = 2, ..., n.$$

This proves the Lemma.

Now we are in position to establish Theorem 1 and Theorem 2. First we prove inequality (1.3). The case $n = 1$ is trivial. Let $n > 1$; we define

$$\mu = 2(1 + \cos t) \qquad \text{with} \quad t = \frac{\pi}{n+1}$$

and

$$\lambda_k = 1 + \frac{\sin(k+1)t}{\sin(kt)}, \qquad k = 1, ..., n.$$

Then we have $\lambda_n = 1$ and since $a_{n+1} = 0$ we conclude from the Lemma

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 + a_n^2 = \sum_{k=1}^{n+1} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos\frac{\pi}{n+1}\right) \sum_{k=1}^{n} a_k^2.$$

Next we establish inequality (1.5). Again the case $n = 1$ is trivial and we may assume $n > 1$. Defining

$$\mu = 2(1 + \cos t) \qquad \text{with} \quad t = \frac{2\pi}{2n+1}$$

and

$$\lambda_k = 1 + \frac{\sin(k+1)t}{\sin(kt)}, \qquad k = 1, ..., n$$

we obtain $\lambda_n = 0$ and from the Lemma we get

$$\sum_{k=1}^{n} (a_k - a_{k-1})^2 \leqslant 2\left(1 + \cos\frac{2\pi}{2n+1}\right) \sum_{k=1}^{n} a_k^2.$$

Finally we have to discuss the cases of equality. Simple calculations reveal that the sign of equality holds in (1.3) and in (1.5) if the a_k's are defined by (1.4) and (1.6), respectively. Now, let us assume that equality is valid in (1.3) or in (1.5). Then we have

$$x_k a_k + y_{k-1} a_{k-1} = 0 \qquad \text{for} \quad k = 2, ..., n,$$

where

$$x_k = [\cos t - \sin t \cot(kt)]^{1/2} \qquad \text{and} \qquad y_{k-1} = \left[\frac{\sin(kt)}{\sin(k-1)t}\right]^{1/2}$$

(with $t = \pi/(n+1)$ or $t = 2\pi/(2n+1)$).

Since $x_k \neq 0$ and $y_{k-1} \neq 0$ for $k = 2, ..., n$, we obtain

$$a_k = -\frac{y_{k-1}}{x_k} a_{k-1} = -(y_{k-1})^2 a_{k-1} \qquad (k = 2, ..., n)$$

which leads to

$$a_k = a_1(-1)^{k-1} \frac{\sin(kt)}{\sin t}.$$

This completes the proof of Theorem 1 and Theorem 2.

3. Concluding Remarks

The paper of Fan, Taussky, and Todd contains also noteworthy inequalities involving the second differences

$$\Delta^2 a_k = a_k - 2a_{k+1} + a_{k+2}.$$

One of their results states that if $a_1, ..., a_n$ are real numbers and if $a_0 = a_{n+1} = 0$, then

$$\left(2 \sin \frac{\pi}{2(n+1)}\right)^4 \sum_{k=1}^{n} a_k^2 \leqslant \sum_{k=0}^{n-1} (a_k - 2a_{k+1} + a_{k+2})^2 \tag{3.1}$$

with equality holding if and only if

$$a_k = c \sin \frac{k\pi}{n+1} \qquad (k = 1, ..., n),$$

where c is a real constant.

We conclude the paper by asking:Does there exist converse inequalities of (3.1) and of related inequalities given in [2], and if the answer is "yes" what are the best possible constants?

References

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